On a new normalization for tractor covariant derivatives

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Abstract

A regular normal parabolic geometry of type G/P on a manifold M gives rise to sequences D_i of invariant differential operators, known as the curved version of the BGG resolution. These sequences are constructed from the normal covariant derivative ∇^ω on the corresponding tractor bundle V, where ω is the normal Cartan connection. The first operator D_0 in the sequence is overdetermined and it is well known that ∇^ω yields the prolongation of this operator in the homogeneous case $M = G/P$. Our first main result is the curved version of such a prolongation. This requires a new normalization $\tilde\nabla$ of the tractor covariant derivative on V. Moreover, we obtain an analogue for higher operators D_i. In that case one needs to modify the exterior covariant derivative d^{∇^ω} by differential terms. Finally we demonstrate these results on simple examples in projective and Grassmannian geometry. Our approach is based on standard techniques of the BGG machinery.

1 Introduction

Let G be a (real) semisimple Lie group and P its parabolic subgroup. Following ideas of É. Cartan, the homogeneous space G/P is a flat model for a curved parabolic geometry of type (G, P), which is specified by a couple $(\mathcal{G}, \omega)$, where $\mathcal{G} \to M$ is a principal P-bundle and ω is a Cartan connection. It is well known that such a geometry can be characterized by an underlying geometric structure on the manifold M, together with a suitable normalization condition (for more

details, see [5]). Distinguished examples of this procedure are the normal Cartan connections constructed for a conformal structure by É. Cartan and for a CR structure by Chern and Moser. Let us consider a regular normal parabolic geometry $(\mathcal{G}, \omega)$ of type (G, P). For any G-module $\mathbb{V}$, the tractor bundle V over M is (by definition) the vector bundle associated to $\mathcal{G}$ and the representation $\mathbb{V}$ (restricted to P). The normal Cartan connection ω on $\mathcal{G}$ then induces the tractor covariant derivative ∇^{ω} on V, which is then used in various problems in analysis and/or geometry on M (e.g., for constructions of differential invariants on the corresponding parabolic geometry). For example, it plays the key role in the construction of the Bernstein-Gel'fand-Gel'fand (BGG) sequences of invariant differential operators (see [6, 4]) and prolongation procedures for first operators in the BGG sequences (see, e.g. [1]).

In particular, there is a lot of interest in the study of properties of the first operators in the BGG sequences, or their semilinear version. Ideas behind the construction of these operators by the BGG machinery can be helpful in such problems. The construction uses tractor covariant derivatives acting on tractor bundles and suitable splitting operators (for details, see Sect. 3). In some simple cases, there is a one-to-one correspondence between solutions of the first BGG equation and the kernel of the corresponding tractor covariant derivative. In other words, the tractor covariant derivative is the prolongation of the first BGG operator. But such a simple correspondence between solutions of the first BGG equation and the kernel of the tractor covariant derivative is not valid in general.

A general scheme for a prolongation of the first BGG operator (and its semilinear version) was introduced in [1], for a generalization to the contact cases, see [9]. The procedure used in [1] is efficient but not invariant. In quite a few special cases (see [2, 14, 10, 11, 7]), several authors found an invariant way how to compute a deformation of the normal tractor covariant derivative having the property that its kernel can be identified with solutions of the first BGG sequence.

The new normalization of tractor covariant derivatives developed in the paper is motivated by a wish to extend these examples to a general scheme. We shall study the problem of a suitable normalization for tractor covariant derivatives for a general parabolic geometry in a systematic way and show that there is a distinguished alternative of the usual normalization of tractor covariant derivatives on tractor bundles giving directly a canonical prolongation of the first BGG operator in an invariant way.

The normal tractor covariant derivative is induced from the normal Cartan connection on the principal bundle $\mathcal{G}$. An important observation is that if we want to find a covariant derivative on tractor bundles giving the invariant prolongation of the first BGG operator, it is necessary to adapt (in contrast to ∇^{ω}) the normalization condition to a choice of the tractor bundle under consideration.

The main results of the paper can be described as follows. Let us consider a regular normal parabolic geometry of type (G, P) given by the couple $(\mathcal{G}, \omega)$. For any irreducible G-module $\mathbb{V}$, there is the associated covariant derivative ∇^{ω} on the associated vector bundle V. The space of all covariant derivatives on V is the affine space modelled on the vector space $\mathcal{E}^1(\operatorname{End} V)$. We want to find a deformation of ∇^{ω} by $\Phi \in \mathcal{E}^1(\operatorname{End} V)$ satisfying a new normalization condition (adapted to the choice of $\mathbb{V}$) in such a way that the resulting covariant derivative

will have suitable properties.

The deformation Φ cannot be chosen arbitrarily. Firstly, the construction of the BGG sequence leads to the requirement to preserve the lowest homogeneous component of ∇^ω (having homogeneity zero), hence we shall restrict to $\Phi \in (\mathcal{E}^1(\text{End } V))^1$, where the superscript 1 indicates that Φ should have the (total) homogeneity bigger or equal to one. The aim to have good properties of the new covariant derivative in the prolongation procedure for the first BGG operator induces further restrictions on a choice of Φ. They will be expressed by properties of values of $\Phi(s) \in \mathcal{E}^1(V)$, where s is a section of V. It leads to the following class of covariant derivatives on the tractor bundle V.

Definition 1.1 *Let ω be the regular normal Cartan connection on the principle bundle $\mathcal{G}$ and let ∇^ω be the associated covariant derivative on the associated vector bundle V. The class $\mathcal{C}$ of admissible covariant derivatives on V is defined by*

$$\mathcal{C} = \left\{ \nabla = \nabla^\omega + \Phi | \Phi \in \text{Im}(\partial_V^* \otimes \text{Id}_{V^*}), \Phi \in (\mathcal{E}^1(\text{End } V))^1, \right\},$$

where ∂_V^* is the Kostant differential corresponding to homology of $\mathfrak{g}_-$ with values in $\mathbb{V}$.

The condition $\Phi \in \text{Im}(\partial_V^* \otimes \text{Id}_{V^*})$ is equivalent to the property $\Phi(s) \in \text{Im } \partial_V^* \subset \mathcal{E}^1(V)$ for all $s \in \Gamma(V)$, where $\Gamma(V)$ denotes the space of sections of V.

The main theorem of the paper is then

Theorem 1.2 *There exists a unique covariant derivative $\nabla \in \mathcal{C}$ with the property*

$$(\partial_V^* \otimes \text{Id}_{V^*})(R^\nabla) = 0,$$

where $R^\nabla \in \mathcal{E}^2(\text{End } V)$ is the curvature of ∇.

Again, the condition $(\partial_V^* \otimes \text{Id}_{V^*})(R^\nabla) = 0$ can be equivalently expressed as the condition $\partial_V^*(R^\nabla(s)) = 0$ for all sections s of V.

The new covariant derivative ∇ constructed in Theorem 1.2 gives a prolongation of the first BGG operator, hence we shall call the covariant derivative satisfying this new normalization condition *the prolongation covariant derivative*. The next main result is the theorem stating this property.

Theorem 1.3 *Let us consider a parabolic geometry $(\mathcal{G}, \omega)$ modeled on a couple (G, P). There is a one-to one correspondence between the kernel of the first BGG operator for a G-module $\mathbb{V}$ and the kernel of the prolongation covariant derivative on the associated bundle V over M.*

In the second part of the paper, we extend the previous construction to other operators in the BGG sequence. In these cases, we have to consider a more general deformation of the exterior derivative d^∇ by adding a differential term (instead of just an algebraic one, which was sufficient for the first operator in the BGG sequence).

Finally, we compare the general procedure developed in the paper with particular results obtained in some special cases and compute some other examples of the prolongation covariant derivatives. They come from projective and Grassmann geometry.

2 Normalization of tractor covariant derivatives

2.1 The double filtration on $\operatorname{End}\mathbb{V}$

Let G be a semisimple Lie group (real or complex) and P its parabolic subgroup. The choice of P induces the grading $\mathfrak{g} = \oplus_{i=-k}^{k}\mathfrak{g}_i$ on the Lie algebra of G. Let $\mathbb{V}$ be an irreducible module for G. There is the grading element E in $\mathfrak{g}_0$ acting by i on $\mathfrak{g}_i$. It can be additively shifted to E' in such a way that eigenvalues of E' on $\mathbb{V}$ are integers between 0 and r for a suitable positive integer r. Eigenvalues of E' on $\mathbb{V}^*$ are then integers between $-r$ and 0. Then we get decompositions of $\mathbb{V}$, resp. $\mathbb{V}^*$, into the corresponding eigenspaces

$$\mathbb{V} = \oplus_{\bar{\iota}=0}^{r}\mathbb{V}_{\bar{\iota}}, \qquad \mathbb{V}^* = \oplus_{\bar{\iota}'=-r}^{0}\mathbb{V}_{\bar{\iota}'}^*.$$

A similar decomposition of $\mathfrak{g}_+$ is given by $\mathfrak{g}_+ = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$.

A bigrading on $\operatorname{End}\mathbb{V} \simeq \mathbb{V} \otimes \mathbb{V}^*$ is then given by

$$\operatorname{End}\mathbb{V} = \oplus_{\bar{\iota}=0}^{r} \oplus_{\bar{\iota}'=-r}^{0} (\mathbb{V}_{\bar{\iota}} \otimes \mathbb{V}_{\bar{\iota}'}^*).$$

Consequently there is a 'diagonal' grading on $\operatorname{End}\mathbb{V}$ given by

$$\operatorname{End}\mathbb{V} = \oplus_{\ell=-r}^{r}(\operatorname{End}\mathbb{V})_\ell; \quad (\operatorname{End}\mathbb{V})_\ell := \oplus_{\bar{\iota}+\bar{\iota}'=\ell}\mathbb{V}_{\bar{\iota}} \otimes \mathbb{V}_{\bar{\iota}'}^*,$$

and 'vertical' (resp. 'horizontal') gradings given by

$$\operatorname{End}\mathbb{V} = \oplus_{\bar{\iota}=0}^{r}(\operatorname{End}\mathbb{V})_{\bar{\iota}}; \quad (\operatorname{End}\mathbb{V})_{\bar{\iota}} := \mathbb{V}_{\bar{\iota}} \otimes \mathbb{V}^*$$

$$\operatorname{End}\mathbb{V} = \oplus_{\bar{\iota}'=-r}^{0}(\operatorname{End}\mathbb{V})_{\bar{\iota}'}; \quad (\operatorname{End}\mathbb{V})_{\bar{\iota}'} := \mathbb{V} \otimes \mathbb{V}_{\bar{\iota}'}^*.$$

The diagonal grading is independent of the normalization of the grading of $\mathbb{V}$. In what follows, we shall use the diagonal and the vertical gradings on $\operatorname{End}\mathbb{V}$.

The gradings are not P-invariant. We shall hence consider filtrations induced by gradings above. For the diagonal grading, we shall define the filtration by a choice of subspaces

$$(\operatorname{End}\mathbb{V})^\ell = \oplus_{k\geq\ell}(\operatorname{End}\mathbb{V})_k,$$

while for the horizontal grading, the filtration is defined by

$$(\operatorname{End}\mathbb{V})^{\bar{\iota}} = \oplus_{\bar{k}\geq\bar{\iota}}(\operatorname{End}\mathbb{V})_{\bar{k}}.$$

The grading of $\mathfrak{g}_+$ also gives the standard filtration $\mathfrak{g}^k \subset \ldots \subset \mathfrak{g}^1 = \mathfrak{g}_+$.

These filtrations (together with the filtration on $\mathfrak{g}_+$) induce also the filtrations on the chain spaces $\Lambda^j(\mathfrak{g}_+) \otimes \operatorname{End}\mathbb{V}$ for the Lie algebra homology and cohomology complexes. The differentials in the Lie algebra (co)homology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$-modules $\mathbb{W}$ are the maps $\partial_{\mathbb{W}} : \Lambda^j(\mathfrak{g}_+) \otimes \mathbb{W} \mapsto \Lambda^{j+1}(\mathfrak{g}_+) \otimes \mathbb{W}$ resp. $\partial_{\mathbb{W}}^* : \Lambda^j(\mathfrak{g}_+) \otimes \mathbb{W} \mapsto \Lambda^{j-1}(\mathfrak{g}_+) \otimes \mathbb{W}$. If $\mathbb{W} = \operatorname{End}\mathbb{V} \simeq \mathbb{V} \otimes \mathbb{V}^*$ for a $\mathfrak{g}$-module $\mathbb{V}$, we shall denote operators $\partial_{\mathbb{V}} \otimes \operatorname{Id}_{\mathbb{V}^*}$, resp. $\partial_{\mathbb{V}}^* \otimes \operatorname{Id}_{\mathbb{V}^*}$, simply by $\partial_{\mathbb{V}}$, resp. $\partial_{\mathbb{V}}^*$. It should not lead to any confusion.

The definition of operators $\partial_{\mathbb{V}}$ and $\partial_{\mathbb{V}}^*$ implies immediately that they preserve both horizontal and diagonal gradings on $\Lambda^j(\mathfrak{g}_+) \otimes \operatorname{End}\mathbb{V}$. Hence they respect both horizontal and diagonal filtrations on $\Lambda^j(\mathfrak{g}_+) \otimes \operatorname{End}\mathbb{V}$. We shall use bellow the induced operators between the graded bundles associated to the horizontal filtration and we shall denote them by $gr\,\partial_{\mathbb{V}}$, resp. $gr\,\partial_{\mathbb{V}}^*$.

2.2   Induced operators on associated graded bundles

The spaces of j-forms on M with values in a bundle W will be denoted by $\mathcal{E}^j(W)$. They are isomorphic to the bundle induced by the P-module $\Lambda^j(\mathfrak{g}_+) \otimes \mathbb{W}$. Similarly, the tangent bundle is isomorphic to the bundle associated to the P-modul $\mathfrak{g}/\mathfrak{p}$. All filtrations mentioned above are P-invariant and they consequently induce the corresponding filtrations on $\mathcal{E}^j(\operatorname{End} V)$. We shall need, in particular, the diagonal filtrations $(\mathcal{E}^j(\operatorname{End} V))^\ell$, resp. the vertical filtration $(\mathcal{E}^j(\operatorname{End} V))^{\bar{\iota}}$, induced on $\mathcal{E}^j(\operatorname{End} V)$. We shall denote by $gr_\ell(\mathcal{E}^j(\operatorname{End} V))$, resp. $gr_{\bar{\iota}}(\mathcal{E}^j(\operatorname{End} V))$ the associated graded bundles.

The operator $gr\partial_{\mathbb{V}}^*$ and $gr\partial_{\mathbb{V}}$ are P-equivariant, hence they induce well defined maps ∂_V^*, resp. ∂_V, between the corresponding associated graded bundles.

We shall denote by $gr\,\partial_V$, resp. $gr\,\partial_V^*$, the direct sum of all maps $gr_{\bar{\iota}}\partial_V$, resp. $gr_{\bar{\iota}}\partial_V^*$ acting on the direct sum $gr\,\mathcal{E}^j(\operatorname{End} V) := \oplus_{\bar{\iota}}gr_{\bar{\iota}}(\mathcal{E}^j(\operatorname{End} V))$. The operators $gr\,\partial_V$, and $gr\,\partial_V^*$ have then usual properties of the Kostant differentials. In particular, they are dual to each other (with respect to a suitable scalar product), which implies usual properties of their kernel and images (the Hodge decomposition).

Note also that $\mathcal{E}^j(V) \otimes V^* = \mathcal{E}^j(\operatorname{End} V)$. Hence the standard filtration on $\mathcal{E}^j(V)$ is transferred (by the tensor product with V^*) to the horizontal grading on $\mathcal{E}^j(\operatorname{End} V)$. As an immediate corollary, we get that $\varphi \in \mathcal{E}^j(\operatorname{End} V)^{\bar{\iota}}$ if and only if $\varphi s \in \mathcal{E}^j(V)^{\bar{\iota}}$ for all sections $s \in \mathcal{E}^0(V)$.

2.3   A choice of normalization

Let us consider a regular parabolic geometry $(\mathcal{G}, \omega)$ over M with the homogeneous model given by a couple (G, P). For an irreducible G-module $\mathbb{V}$, we shall consider the associated tractor bundle V on M. The curvature κ of the Cartan connection ω is a two-form with values in the adjoint tractor bundle $\mathcal{A} \simeq \mathcal{G} \times_P \mathfrak{g}$. The usual normalization condition for ω, expressed in terms of the Kostant differential ∂^* corresponding to homology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$, requires the curvature κ to be ∂^*-closed. In terms of an associated covariant derivative ∇^ω on V, the curvature R^{∇^ω} of ∇^ω is a two-form with values in $\operatorname{End} V$ and the normalization condition can be expressed using the Kostant differential ∂^* for $\operatorname{End} V$ as

$$\partial^*(R^{\nabla^\omega}) = 0.$$

Given a choice of the bundle V, we are going to change the normalization condition for a covariant derivative ∇ on V. Let Id_{V^*} denote the identity map on V^*. As above in the algebraic version, we shall consider operators

$$\partial_V \otimes \operatorname{Id}_{V^*}, \; \partial_V^* \otimes \operatorname{Id}_{V^*}.$$

acting on forms $\mathcal{E}^j(\operatorname{End} V)$ with values in $\operatorname{End} V \simeq V \otimes V^*$. Abusing the notation, we shall denote them by ∂_V, resp. ∂_V^*. It will always be clear whether the differentials act on forms with values in V or forms with values in $\operatorname{End} V$.

We shall now introduce a new normalization for covariant derivatives on V.

Definition 2.1 *We shall call a covariant derivative $\nabla \in \mathcal{C}$ the prolongation covariant derivative, if*

$$\partial_V^*(R^\nabla) = 0,$$

where $R^\nabla \in \mathcal{E}^2(\operatorname{End} V)$ is the curvature of ∇.

The choice of its name should suggest that the new normalization condition gives better properties to ∇ in the prolongation procedure for the first operator in the BGG sequence corresponding to the representation $\mathbb{V}$ (or its semilinear versions).

We shall need the following property.

Lemma 2.2 *If $\varphi \in (\mathcal{E}^1(\operatorname{End} V))^{\bar{\imath}}$ and $\tau \in \mathcal{E}^1(V)$, then*

$$\varphi \wedge \tau \in (\mathcal{E}^2(V))^{\bar{\imath}+1}.$$

Proof.

Indeed, we can decompose φ into homogeneous components

$$\varphi = \sum_j \alpha_j \otimes v_j \otimes w_j, \alpha_j \in \mathcal{E}^1, v_j \in V, w_j \in V^*,$$

where the sum of homogeneities of α_j and v_j is greater or equal to $\bar{\imath}$. If we also decompose τ as

$$\tau = \sum_k \beta_k \otimes u_k, \beta_k \in \mathcal{E}^1, u_k \in V,$$

then the expression

$$\varphi \wedge \tau = \sum_{j,k} w_j(u_k)\alpha_j \wedge \beta_k \otimes v_j$$

clearly has summands of homogeneity greater or equal to $\bar{\imath}+1$. $\qquad\square$

2.4 The main lemma.

The key information for the normalization procedure is the following fact concerning the induced change of the curvature.

Lemma 2.3 *Let ∇_1, resp. ∇_2, be two covariant derivatives from $\mathcal{C}$ related to each other by the deformation $\Phi = \nabla_2 - \nabla_1 \in (\mathcal{E}^1(\operatorname{End} V))^1$ and let R_1, resp. R_2, be the corresponding curvatures.*
If $\Phi \in (\mathcal{E}^1(\operatorname{End} V))^{\bar{\imath}}$, then $R_2 - R_1 \in (\mathcal{E}^2(\operatorname{End} V))^{\bar{\imath}}$ and

$$gr_{\bar{\imath}}(R_2 - R_1) = (gr_{\bar{\imath}}\,\partial_V)(gr_{\bar{\imath}}\,\Phi).$$

Proof.

Let ω be the normal Cartan connection for the chosen parabolic geometry and ∇ its associated covariant derivative. It is well known that ∇ and d^∇ preserve the standard filtration on $\mathcal{E}^j(V)$ and that the corresponding graded version of ∇, resp. d^∇ is equal to $gr\,\partial_V$. A shift of ∇ by $\Phi \in (\mathcal{E}^1(\operatorname{End} V))^1$ does not change this property, the same being true for $d^{\nabla+\Phi}$.

The change in the curvature is then

$$R_2 - R_1 = d^\nabla \Phi + [\Phi, \Phi].$$

The result clearly belongs to $\mathcal{E}^2(\operatorname{End} V)^{\bar{\imath}}$, because the operator d^∇ preserves the filtrations and we can use Lemma 2.2 for the second term.

Then we get for any $s \in \mathcal{E}^0(V)$,

$$gr_{\bar{\iota}}((d^\nabla \Phi + [\Phi, \Phi])s) = gr_{\bar{\iota}}((d^\nabla \Phi)s) =$$

$$= gr_{\bar{\iota}}(d^\nabla(\Phi s) - \Phi \wedge (\nabla s)) = gr_{\bar{\iota}}(\partial_V(\Phi(s))) =$$

$$= (gr_{\bar{\iota}}\partial_V)(gr_{\bar{\iota}}(\Phi(s))).$$

$\square$

2.5 Existence

Lemma 2.4 *Suppose that there is a tractor covariant derivative $\nabla \in \mathcal{C}$ with the property*

$$\partial_V^*(R^\nabla) \in \mathcal{E}^1(\text{End } V)^{\bar{\iota}},$$

where $\bar{\iota}$ is a number between 0 and r.

Then there exists $\Phi \in \mathcal{E}^1(\text{End } V)^1 \cap \mathcal{E}^1(\text{End } V)^{\bar{\iota}}$ such that for $\tilde{\nabla} = \nabla + \Phi$, we have

$$\partial_V^*(R^{\tilde{\nabla}}) \in \mathcal{E}^1(\text{End } V)^{\bar{\iota}+1}.$$

Proof.

The spaces $\{\mathcal{E}^1(\text{End } V)^1 \cap \mathcal{E}^1(\text{End } V)^{\bar{\iota}}\}_{\bar{\iota}=0}^r$ give a descending filtration of the space $\mathcal{E}^1(\text{End } V)^1$. The filtration is preserved by maps ∂_V and ∂_V^*, hence they induce maps on the associated graded bundle (we denote them for simplicity of notation by the same symbols as for the full filtration of $\mathcal{E}^1(\text{End } V)$). The standard Kostant decomposition says that $\text{Ker } gr\,\partial_V^*$ and $\text{Im } gr\,\partial_V$ are complementary subspaces of the graded bundle $gr\,\mathcal{E}^1(\text{End } V)^1$. In particular, $gr\,\partial_V^*$ restricts to an isomorphism of $\text{Im } gr\,\partial_V$ to $\text{Im } gr\,\partial_V^*$.

Hence there is an element $\varphi \in gr_{\bar{\iota}}(\mathcal{E}^1(\text{End } V)^1)$ such that

$$(gr\,\partial_V^*)((gr\,\partial_V)(\varphi)) = gr_{\bar{\iota}}(\partial_V^*(R^\nabla)).$$

We shall take any preimage $\Phi \in \mathcal{E}^1(\text{End } V)^1 \cap \mathcal{E}^1(\text{End } V)^{\bar{\iota}}$ of φ and we shall define a corrected covariant derivative by $\tilde{\nabla} = \nabla - \Phi$.

Due to Lemma 2.3, we get

$$\begin{aligned} gr_{\bar{\iota}}(\partial_V^*(R^{\tilde{\nabla}})) &= gr_{\bar{\iota}}(\partial_V^*(R^\nabla)) - (gr\,\partial_V^*)(gr_{\bar{\iota}}(R^\nabla - R^{\tilde{\nabla}})) = \\ &= gr_{\bar{\iota}}(\partial_V^*(R^\nabla)) - (gr\,\partial_V^*)((gr\,\partial_V)(gr_{\bar{\iota}}(\Phi))) = 0. \end{aligned}$$

Hence $\tilde{\nabla}$ has the required properties.

$\square$

Theorem 2.5 *For each irreducible G-module $\mathbb{V}$, there exists a prolongation covariant derivative $\nabla \in \mathcal{C}$, i.e., we can find $\nabla \in \mathcal{C}$ such that*

$$\partial_V^*(R^\nabla) = 0.$$

Proof.

The curvature function of the regular normal connection ω for the corresponding parabolic geometry belongs (by definition of regularity) to $\mathcal{E}^2(\mathcal{A})^1$, so $R^{\nabla^\omega} \in \mathcal{E}^2(\text{End } V)^1$, and $\partial_V^*(R^{\nabla^\omega}) \in \mathcal{E}^1(\text{End } V)^1$. Using Lemma 2.4., we get (by induction) the claim of the theorem.

$\square$

2.6 Uniqueness.

Theorem 2.6 *Suppose that ∇_1, ∇_2 are two covariant derivatives in $\mathcal{C}$, both satisfying the normalization condition $\partial_V^*(R^\nabla) = 0$. Then*

$$\nabla_1 = \nabla_2.$$

Proof.

Let $\Phi_1, \Phi_2 \in \mathcal{E}^1(\operatorname{End} V)^1 \cap \operatorname{Im} \partial_V^*$ such that

$$\nabla_1 = \nabla^\omega + \Phi_1; \ \nabla_2 = \nabla^\omega + \Phi_2.$$

Denote by R_1, resp. R_2, the curvatures of ∇_1, resp. ∇_2. Then $\Phi = \Phi_2 - \Phi_1$ belongs to $\mathcal{E}^1(\operatorname{End} V)^1 \cap \operatorname{Im} \partial_V^*$. Suppose now that $\Phi \in \mathcal{E}^1(\operatorname{End} V)^{\bar{\imath}}$. By assumption, $gr_{\bar{\imath}}(R_2 - R_1)$ is in the kernel of $gr\,\partial_V^*$. By Lemma 2.3, we have

$$gr_{\bar{\imath}}(R_2 - R_1) = (gr\,\partial_V)(gr_{\bar{\imath}}\,\Phi).$$

But $\operatorname{Ker} gr\,\partial_V^* \cap \operatorname{Im} gr\,\partial_V$ is trivial hence $gr_{\bar{\imath}}(R_2 - R_1) = 0$. Hence $gr_{\bar{\imath}}\,\Phi$ is in the kernel of $gr\,\partial_V$, and also in the image of $gr\,\partial_V^*$, by assumption. Hence $gr_{\bar{\imath}}\,\Phi = 0$. By induction, $\Phi = 0$. $\qquad\square$

The construction above depends on some choices (e.g., a choice of a preimage Φ of φ). Nevertheless, the uniqueness of the prolongation covariant derivative shows that the result of the construction is independent of all choices. Hence we get the following corollary.

Corollary 2.7 *The prolongation covariant derivative is invariant. This means that it depends only on the data of the chosen parabolic structure and the bundle V.*

3 The prolongation of the first BGG operator.

The BGG complexes are sequences of invariant differential operators on a homogeneous model for a given parabolic geometry. A curved version of it, i.e., an extension of operators in the sequence to invariant differential operators on general (non-flat) manifolds with a given parabolic structure was first constructed in [6] and the construction was simplified and extended in [4]. The first operator in such a sequence always gives an overdetermined system of invariant differential equations. A prolongation of this operator was constructed for the case of 1-graded parabolic geometries in [1]. However, the methods used there needed a choice of the Weyl structure, hence the resulting covariant derivative was not invariant. We are now going to show that the normalization of tractor connections described in the paper can be used to obtain invariant (natural) prolongations.

We begin by introducing the setting and basic operators of the BGG-machinery in a generalized version needed for the next section. Let V be a tractor bundle over M with a covariant derivative ∇ and the exterior covariant derivative $d^\nabla : \mathcal{E}^k(V) \mapsto \mathcal{E}^{k+1}(V)$. Recall from above that we have a well defined differential $\partial^* = \partial_V^* : \mathcal{E}^{k+1}(V) \to \mathcal{E}^k(V)$. The property $\partial^* \circ \partial^* = 0$ allows us to define the cohomology H_k as the vector bundle quotient $H_k = \operatorname{Ker} \partial^* / \operatorname{Im} \partial^*$, where $\operatorname{Ker} \partial^* \subset \mathcal{E}^k(V)$ is the space of cycles and $\operatorname{Im} \partial^* \subset \mathcal{E}^k(V)$ is the space of

boundaries. The canonical surjection $\operatorname{Ker}\partial_V^* \subset \mathcal{E}^k(V) \mapsto H_k$ will be denoted by Π_k.

Due to regularity of the parabolic geometry under consideration, the operators d^∇ are homogeneous of degree zero with respect to the natural filtration of the spaces $\mathcal{E}^k(V)$ and they induce the algebraic differential $gr\partial_V : gr(\mathcal{E}^k(V)) \mapsto gr(\mathcal{E}^{k+1}(V))$ on the associated graded spaces. Thus it is possible to regard d^∇ as a natural lift of $gr(\partial_V)$ to a differential operator from $\mathcal{E}^k(V)$ to $\mathcal{E}^{k+1}(V)$.

The main ingredients in the BGG-machinery are the differential splitting operators $L_k : H_k \mapsto \operatorname{Ker}\partial_V^* \subset \mathcal{E}^k(V)$ with the property $\partial^* \circ d^\nabla \circ L_k = 0$. This allows one to define the BGG-operators $D_k : H_k \mapsto H_{k+1}$ in the obvious way: $D_k := \Pi_k \circ d^\nabla \circ L_k$. The definition is encoded in the diagram

$$
\begin{array}{ccc}
\mathcal{E}^k(V) & \xrightarrow{\ d^\nabla\ } & \mathcal{E}^{k+1}(V) \\
{\scriptstyle i}\big\uparrow & & {\scriptstyle i}\big\uparrow \\
\operatorname{Ker}\partial^* & & \operatorname{Ker}\partial^* \\
{\scriptstyle L_k}\big\uparrow & & \big\downarrow{\scriptstyle \Pi_{k+1}} \\
H_k & \xrightarrow{\ D_k\ } & H_{k+1}
\end{array}
\tag{1}
$$

where i denotes the inclusion.

We shall introduce the construction of the splitting operators in a more general situation, where the exterior covariant derivatives d^∇ on $\mathcal{E}^k(V)$ will be substituted by general differential operators E_k with suitable properties (see the theorem below). The operators D_k are defined by the same construction as the BGG operators and they depend, in general, on the choice of E_k. The theorem below shows that for certain classes of operators E_k, the resulting operators L_k and D_k do not change.

Theorem 3.1 *Let $(\mathcal{E}^k(V))^j$ denote the filtration on $\mathcal{E}^k(V)$ and let $gr(\mathcal{E}^k(V))$ denote the associated graded bundle, similarly for $\mathcal{E}^{k+1}(V)$. Let E_k be a filtration preserving differential operator from $\mathcal{E}^k(V)$ to $\mathcal{E}^{k+1}(V)$ with the property that the associated graded map coincides with $gr\,\partial$.*

Then for every $\sigma \in H_k$, there exists a unique element $s \in \operatorname{Ker}\partial^*$ with the following properties:

(1) $\Pi_k(s) = \sigma$,

(2) $E_k(s) \in \operatorname{Ker}\partial^*$.

Moreover, the mapping L_k defined by $\sigma \mapsto L_k(\sigma) := s$ is given by a differential operator. The corresponding operator D_k is then defined by

$$D_k := \Pi_{k+1} \circ E_k \circ L_k : H_k \mapsto H_{k+1}.$$

Suppose that we change the operator E_k to $\tilde{E}_k = E_k + \Phi_k$, where the map $\Phi_k : \mathcal{E}_k(V) \to \mathcal{E}_{k+1}(V)$ is a differential operator with values in $\operatorname{Im}\partial^*$, and preserving the filtration with the property that the associated graded map is trivial.

Then the construction does not change the splitting operator L_k and the operator D_k.

Proof.

The first part of the proof follows the standard line of arguments. The operator $\partial^* \circ E_k$ acts on $\mathcal{E}^k(V)$ and it preserves $\operatorname{Im}\partial^*$. It preserves the filtration

and its graded version is, by assumption, given by $gr(\partial^*) \circ gr(\partial)$, which is invertible on $\operatorname{Im}\partial^*$. Hence also $\partial^* \circ E_k$ is invertible on $\operatorname{Im}\partial^*$ and it is possible to show that its inverse Q is a differential operator.

We can then define a differential operator $\hat{L}_k := \operatorname{Id} - Q \circ \partial^* \circ E_k$, which restricts to zero on $\operatorname{Im}\partial^*$. Hence it induces a well-defined differential operator L_k from H_k to $\operatorname{Ker}\partial^* \subset \mathcal{E}^k(V)$. It is easy to check that the operator L_k satisfies three properties

$$\operatorname{Im} L_k \subset \operatorname{Ker}\partial^*,\ \Pi_k \circ L_k = \operatorname{Id},\ \partial^* \circ E_k \circ L_k = 0.$$

To show that L_k is uniquely characterized by these properties, let us consider $s_1, s_2 \in \operatorname{Ker}\partial^*$ such that $E_k(s_i) \in \operatorname{Ker}\partial^*$, $i = 1, 2$ and $\Pi_k(s_1) = \Pi_k(s_2)$. Then the difference $s = s_1 - s_2$ belongs to $\operatorname{Im}\partial^*$. By definition of $\hat{L}_k$, the relation $\partial^* \circ E_k(s) = 0$ implies $\hat{L}_k(s) = s$. On the other hand, $\hat{L}_k$ is trivial on $\operatorname{Im}\partial^*$, Hence $\hat{L}_k(s) = 0$.

To prove the last statement of the theorem, we shall consider a section s of $\mathcal{E}^k(V)$. The new operator $\tilde{E}_k$ preserves the filtration and the induced graded map is still $gr\,\partial$. Since $(\tilde{E}_k - E_k)s$ belongs to $\operatorname{Im}\partial_V^*$, one has $\tilde{E}_k(s) \in \operatorname{Ker}\partial_V^*$ iff $E_k(s) \in \operatorname{Ker}\partial_V^*$, which shows that $\tilde{L}_k = L_k$. Thus, for $\sigma \in H_k$, one has $(\tilde{E}_k\tilde{L}_k - E_k L_k)\sigma \in \operatorname{Im}\partial_V^*$, but this lies in the kernel of the projection $\Pi_{k+1} : \operatorname{Ker}\partial^* \mapsto H_{k+1}$. $\qquad\square$

Now we want to discuss the relation between $\operatorname{Ker} E_k$ and $\operatorname{Ker} D_k$. For that, we have to consider two consecutive operators E_k and E_{k+1} at the same time. They define two splitting operators L_k and L_{k+1}. We get in such a way the diagram

$$\begin{array}{ccc}
\mathcal{E}^k(V) & \xrightarrow{\ E_k\ } & \mathcal{E}^{k+1}(V) \\
{\scriptstyle L_k}\big\uparrow & & \big\uparrow{\scriptstyle L_{k+1}} \\
H_k & \xrightarrow{\ D_k\ } & H_{k+1}
\end{array} \qquad (2)$$

which, in general, does not commute but there is a convenient criterion for its commutativity.

Theorem 3.2 *The diagram (2) commutes if and only if $\partial^* \circ E_{k+1} \circ E_k(s) = 0$ for all sections $s \in \operatorname{Im} L_k \subset \mathcal{E}^k(V)$.*

Proof.

The values of L_k are uniquely characterized by the conditions $L_k(\sigma) \in \operatorname{Ker}\partial^*$ and $E_k \circ L_k(\sigma) \in \operatorname{Ker}\partial^*$. Similarly, the values of L_{k+1} are characterized by $L_{k+1}(\tau) \in \operatorname{Ker}\partial^*$ and $E_{k+1} \circ L_{k+1}(\tau) \in \operatorname{Ker}\partial^*$. Hence $E_k \circ L_k(\sigma) = L_{k+1} \circ D_k(\sigma)$ iff $E_{k+1} \circ E_k \circ L_k(\sigma) \in \operatorname{Ker}\partial^*$ for all $\sigma \in H_k$. $\qquad\square$

If the diagram above is commutative, we get immediately a one-to-one correspondence between $\operatorname{Ker} E_k \cap \operatorname{Ker}\partial^*$ and $\operatorname{Ker} D_k$.

Theorem 3.3 *Suppose that the diagram (3) commutes. Then Π_k and L_k restrict to inverse isomorphisms between $\operatorname{Ker} E_k \cap \operatorname{Ker}\partial^*$ and $\operatorname{Ker} D_k$.*

Proof. Let s be in $\operatorname{Ker} E_k \cap \operatorname{Ker}\partial^*$. Then $s = L_k(\Pi_k(s))$ by definition of L_k, and $\Pi_k(s) \in \operatorname{Ker} D_k$ by definition of D_k.

On the other hand, if $D_k(\sigma) = 0$, then commutativity of the diagram implies that also

$$L_{k+1} \circ D_k(\sigma) = E_k \circ L_k(\sigma) = 0,$$

hence $L_k \in \operatorname{Ker} E_k \cap \operatorname{Ker} \partial^*$.

And by definition of L_k, we have $\Pi_k \circ L_k = \operatorname{Id}$. $\qquad\qquad\square$

Now we can return back to properties of the prolongation covariant derivative ∇ on V. Using the above claims in the special case of the first square and operators $E_0 = \nabla$ and $E_1 = d^\nabla$, we see immediately that $E_1 \circ E_0 = R^\nabla$. Hence we get the following corollary.

Corollary 3.1 *Consider a tractor bundle V and the corresponding prolongation covariant derivative ∇. Set $E_0 = \nabla$ and $E_1 = d^\nabla$.*

Then the square constructed using these two operators commute and the covariant derivative ∇ gives a prolongation of the first BGG operator D_0. In particular, the splitting operator L_0 induces a one-to-one correspondence between the space of parallel sections of V with respect to ∇ and the kernel of the first BGG operator D_0.

Remark. In the case of a 1-graded geometry, it was shown in [1] that the map $L_0 : H_0 \mapsto V$ induces an isomorphism of $J^k H_0$ with $V_{\leq k}$ for every k such that the homology of $H_1(\mathfrak{g}_-, \mathbb{V})$ sits in homogeneity $> k$. Thus, for every operator $\tilde{D}_0 : H_0 \mapsto H_1$ which differs from the standard BGG-operator D_0 by a linear differential operator of order $\leq k$, there is a map $\Psi \in \mathcal{E}^1(\operatorname{End} V)$ with values in $\operatorname{Ker} \partial_V^*$ such that its induced first BGG-operator coincides with $\tilde{D}_0$. The mapping Ψ is unique up to maps $\mathcal{E}^1(\operatorname{End} V)$ with values in $\operatorname{Im} \partial_V^*$, and it is thus easy to see that therefore the resulting normalized connection $\tilde{\nabla} = \nabla + \Psi + \Phi$ doesn't depend on the choice of Ψ. Thus, natural deformations of D_0 of low enough order can be prolonged naturally as well. We remark that a similar procedure works in the case of general graded parabolic geometries, where one has to use the filtration of the manifold for a suitable version of jet bundles.

4 Prolongation covariant derivatives for the whole BGG sequence

In this section we shall treat the problem considered above in the case of other squares of the BGG sequence. We want to deform the exterior covariant derivative d^∇ on k-forms in such a way that all squares in the generalized BGG construction will commute, and, at the same time, the BGG operators D_k will not change. In fact, we shall succeed to keep both the BGG operators D_k and the splitting operators L_k unchanged. The deformation of d^∇ on $\mathcal{E}^k(V)$ will have, however, a different character. It will be replaced by $E_k := d^\nabla + \Phi_k$, where Φ_k is a linear differential operator mapping $\mathcal{E}^k(V)$ to $\mathcal{E}^{k+1}(V)$. Hence the deformation Φ_k will not be, in general, algebraic. Necessary tools were already prepared in the previous section (Theorems 3.1.- 3.3.). Methods described in this section can also be applied to the first square but they give different answer (and also in this case the deformation Φ_0 will not be algebraic in general).

To describe allowed deformations of the exterior derivative d^∇, we shall introduce the following notation. There are two different filtrations on the

space $A := \operatorname{Hom}(\mathcal{E}^k(V), \mathcal{E}^{k+1}(V))$. The diagonal filtration A^j is induced by the standard filtration on $\mathcal{E}^k(V)$, which is defined by the condition $\Phi(s) \in (\mathcal{E}^{k+1}(V))^{a+j}$ for all $s \in (\mathcal{E}^k(V))^a$. The other (vertical) filtration $A^{\bar\iota}$ is defined by the condition $\Phi(s) \in (\mathcal{E}^{k+1}(V))^{\bar\iota}$ for all $s \in \mathcal{E}^k(V)$. In this paragraph, we shall use symbols ∂ and ∂^* for the Kostant differential associated to the spaces $\mathcal{E}^k(V)$. Recall that the class $\mathcal{C}$ of admissible covariant derivatives on V was defined by

$$\mathcal{C} = \left\{ \nabla = \nabla^\omega + \Phi | \Phi \in \operatorname{Im}(\partial_V^* \otimes \operatorname{Id}_{V^*}), \Phi \in (\mathcal{E}^1(\operatorname{End} V))^1 \right\}.$$

We shall consider the following spaces $\mathcal{C}_k$ of deformations.

Definition 4.1 *The space of allowed deformations will be defined by*

$$\mathcal{C}_k := \{ E_k \in \operatorname{Hom}(\mathcal{E}^k(V), \mathcal{E}^{k+1}(V)) | E_k = d^\nabla + \Phi, \ \Phi \in A^1, \operatorname{Im} \Phi \subset \operatorname{Im} \partial^* \}$$

Theorem 4.2 *(1) Let ∇ be any covariant derivative from $\mathcal{C}$. Let us consider the BGG sequence with the splitting operators L_k and the BGG operators $\{D_k\}$ induced (via Theorem 3.1) by operators $E_k = d^\nabla$*

$$
\begin{array}{ccc}
\mathcal{E}^k(V) & \xrightarrow{\ d^\nabla\ } & \mathcal{E}^{k+1}(V) \\
L_k \uparrow & & \uparrow L_{k+1} \\
H_k & \xrightarrow{\ D_k\ } & H_{k+1}
\end{array}
\tag{3}
$$

Then there exists a collection of differential operators $\Phi_k \in \mathcal{C}_k$ such that $\partial^* \circ d^\nabla \circ (d^\nabla + \Phi_k) = 0$. Moreover, the collection Φ_k with these properties is unique.

(2) As a consequence, the diagrams

$$
\begin{array}{ccc}
\mathcal{E}^k(V) & \xrightarrow{\ d^\nabla + \Phi_k\ } & \mathcal{E}^{k+1}(V) \\
L_k \uparrow & & \uparrow L_{k+1} \\
H_k(V) & \xrightarrow{\ D_k\ } & H_{k+1}(V)
\end{array}
\tag{4}
$$

commute for all $k = 0, 1, \ldots, n-1$.

If ∇ depends only on data of the chosen parabolic geometry, the same is true for operators $E_k = d^\nabla + \Phi_k$.

Proof. Let us choose $k = 0, \ldots, n-1$ and consider the square (4) in the generalized BGG sequence constructed using operators d^∇, where ∇ is any covariant derivative from $\mathcal{C}$. We shall first prove the first assertion of the theorem.

The spaces $\{A^1 \cap A^{\bar\iota}\}_{\bar\iota=0}^r$ form a decreasing filtration of the space A^1 with $\bar\iota = 0, \ldots, r$. The filtration is preserved by maps ∂_V and ∂_V^*, hence they induce maps on the associated graded bundle (we denote them for simplicity of notation by the same symbols as for the full filtration of A). We can consider the Kostant Laplacian $\Box = gr\,\partial_V^*\, gr\,\partial_V + gr\,\partial_V\, gr\,\partial_V^*$. The standard Kostant decomposition says that $\operatorname{Ker}\Box$, $\operatorname{Im} gr\,\partial_V^*$ and $\operatorname{Im} gr\,\partial_V$ are complementary subspaces of the graded bundle $gr\,\mathcal{E}^i(V)^1$. In particular, $\Box$ is invertible on $\operatorname{Im} gr\,\partial_V^*$.

Let us consider two consecutive squares with operators $E_k = d^\nabla$ and $E_{k+1} = d^\nabla$. We know that the operator $G := \partial^* \circ E_{k+1} \circ E_k = \partial_V^*(R^{d^\nabla})$ belongs to A^1 and that the k-th square is commutative iff $G = 0$. If it is not the case, we shall consider the maximal index $\bar\iota = 0$ with the property that $G \in A^{\bar\iota}$.

The map $\Phi^{(1)} = -\Box^{-1} gr(G)$ can be lifted to a linear algebraic map $\Phi^{(1)} : \mathcal{E}^k(V) \mapsto \mathcal{E}^{k+1}(V)$ (e.g., by a choice of the Weyl structure) and we shall define the first iteration $E_k^{(1)} = d^\nabla + \Phi^{(1)}$. Note that the lowest homogeneous component of $E_k^{(1)}$ remains to be ∂_V and that the image of $E_k^{(1)}$ is a subset of $\operatorname{Im} \partial^*$.

Due to
$$E_{k+1} \circ E_k^{(1)} - E_{k+1} \circ E_k = d^\nabla \circ \Phi^{(1)},$$
we get
$$
\begin{aligned}
gr_{\bar\iota}(\partial^* \circ E_{k+1} \circ E_k^{(1)})) &= gr_{\bar\iota}(G + \partial^* \circ d^\nabla \circ \Phi^{(1)}) = \\
&= gr_{\bar\iota}(G) - (gr\,\partial^*)(gr\,\partial_V)(\Box^{-1}(gr_{\bar\iota}(G))) = 0.
\end{aligned}
$$

Hence the first order differential operator $G^{(1)} := \partial^* \circ E_{k+1} \circ E_k^{(1)}$ belongs to $A^{\bar\iota+1}$.

The same procedure will be repeated inductively. If we define
$$\Phi^{(2)} = -(gr\,\partial^*)\Box^{-1}gr_{\bar\iota+1}(G^{(1)})$$
we can again lift this first order differential operator to a first order differential operator $\Phi^{(2)} : \mathcal{E}^k(V) \mapsto \mathcal{E}^{k+1}(V)$ and we can define the next iteration by
$$E_k^{(2)} := E_k^{(1)} + \Phi^{(2)}.$$

Then we get
$$
\begin{aligned}
gr_{\bar\iota}(\partial_V^* \circ d^\nabla \circ E_k^{(2)})) &= gr_{\bar\iota}(G^{(1)} + \partial_V^* \circ d^\nabla \circ \Phi^{(2)}) = \\
&= gr_{\bar\iota}(G^{(1)}) - (gr\,\partial_V^*)(gr\,\partial_V)(\Box^{-1}(gr_{\bar\iota}(G^{(1)}))) = 0.
\end{aligned}
$$

Hence the first order differential operator $G^{(2)} := \partial_V^* \circ d^\nabla \circ E_k^{(2)}$ belongs to $A^{\bar\iota+2}$.

It is clear that by a finite number of iterations, we shall get the existence part of the theorem.

The proof of the uniqueness part of the theorem is similar to the case of Theorem 2.6. Suppose that we have two differential operators Φ_k' and Φ_k'' satisfying the conditions of the theorem. Their difference $\Phi = \Phi_k' - \Phi_k''$ satisfies $\partial_V^*(d^\nabla \circ \Phi) = 0$. To show that $\Phi = 0$, suppose that Φ is nontrivial and consider the biggest $\bar\iota$ such that $(\Phi)^{\bar\iota}$ is nontrivial. Then we know that $gr_{\bar\iota}(d^\nabla \circ \Phi) = (gr\,\partial_V)(gr_{\bar\iota}(\Phi))$, hence $(gr\,\partial_V)(gr_{\bar\iota}(\Phi))$ is at the same time in $\operatorname{Im} gr\,\partial_V$ and $\operatorname{Ker} gr\,\partial_V^*$, hence $(gr\,\partial_V)(gr_{\bar\iota}(\Phi)) = 0$. By definition, $gr_{\bar\iota}(\Phi)$ belongs also to $\operatorname{Im} \partial_V^*$, hence $gr_{\bar\iota}(\Phi)$ is trivial and we have a contradiction.

As for the second part of the theorem, let us consider two consecutive squares in the BGG construction induced by $E_k = d^\nabla$, containing operators D_k and D_{k+1}. If Φ_k is the deformation constructed above, then the replacement of $E_k = d^\nabla$ by $\tilde{E}_k = d^\nabla + \Phi_k$ leads to the same splitting operator L_k. Hence by the first part of the theorem, the k-th diagram commutes. Note that the change of the next operator E_{k+1} will not change the splitting operator L_{k+1}, hence the commutativity of the k-th diagram is preserved.

Finally, during the construction there were several choices made but due to the uniqueness of the result, the construction depends only on data of the chosen parabolic geometry. The same is true for the covariant derivative ∇.

$\square$

5   Examples.

We want to illustrate in this section general results presented above by explicit examples showing a form of the prolongation covariant derivative in some simple situations. Some basic examples in conformal geometry can be found in [16]. A more comprehensive set of examples will be treated in [18].

To calculate the prolongation connection of the first BGG-operator D_0 for some tractor bundle $V = \mathcal{G} \times_P \mathbb{V}$, we employ the theory of Weyl structures [3], [5]. Both of our examples below will be |1|-graded parabolic geometries, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Modding out $P_+ \cong \mathfrak{g}_1$ of the parabolic structure bundle $\mathcal{G}$, one obtains $\mathcal{G}_0 := \mathcal{G}/P_+$, which is a G_0-principal bundle over M. A splitting $\sigma : \mathcal{G}_0 \mapsto \mathcal{G}$ of the canonical projection $\mathcal{G} \mapsto \mathcal{G}_0$ is called a *Weyl structure*, and for our geometric structures below this can be identified with the choice of a *Weyl connection*, which is a linear connection D compatible with the geometry. Under such a choice, all P-associated bundles reduce to G_0-associated bundles, and in particular one gets a decomposition of the tractor bundle V which depends on the choice of Weyl structure. The adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ decomposes into $\mathcal{A}_- M \oplus \mathcal{A}_0 M \oplus \mathcal{A}_1 M$, with $\mathcal{A}_- M \cong TM$ and $A_+ M \cong T^*M$. The Lie algebraic action of $\mathfrak{g}$ on $\mathbb{V}$ gives rise to an action $\bullet$ of $\mathcal{A}M$ on V, which we can restrict to TM and T^*M. The tractor connection ∇^ω can be written as $\nabla^\omega = \partial + D + \mathsf{P}\bullet$: the map $\partial : V \mapsto \Omega^1(M,V)$ is obtained by the action of $TM \hookrightarrow \mathcal{A}M$ on V, and $\mathsf{P}\bullet : V \mapsto \Omega^1(M,V)$ is induced by the action of the second slot of the (generalized) Schouten tensor $\mathsf{P} \in \mathcal{E}_{ab}$ of D, which will be symmetric for our choices of D. Recall that this decomposition of ∇^ω depends on the choice of Weyl structure $\sigma : \mathcal{G}_0 \mapsto \mathcal{G}$ resp. Weyl connection D.

In our explicit formulas, we employ abstract index notation [19]: $\mathcal{E}_a = \Omega^1(M), \mathcal{E}^a = \mathfrak{X}(M)$, multiple indices are tensor products. Round brackets denote symmetrizations of the enclosed indices, square brackets denote skew symmetrizations. A subscript zero takes the trace-free part.

We are now going to prolong an interesting equation in projective geometry which has already been treated in [10] by different methods, and another equation for Grassmannian structures of type $(2,q), q > 2$. For a more detailed exposition of explicit calculations cf. [17],[16] and the forthcoming [18].

5.1   An example in projective geometry

Let M be an orientable manifold of dimension n endowed with a projective class of linear, torsion-free connections $[D]$; here D and D' are projectively equivalent if there is a $\Upsilon \in \mathcal{E}^1$ such that

$$D'_a \omega_b = D_a \omega - \Upsilon_a \omega_b - \Upsilon_b \omega_a,$$

see e.g. [8]. For simplicity, we will assume that our chosen representatives $D \in [D]$ preserve a volume form on TM.

To define projectively invariant operators we need to employ the *projective densities*, which are line bundles $\mathcal{E}[w], w \in \mathbb{R}$ associated to the full GL(n)-frame bundle of TM via the 1-dimensional representation

$$C \in \mathrm{GL}(n) \mapsto |\det C|^{w\frac{n+1}{n}} \in \mathbb{R}_+.$$

We are going to prolong the following projectively invariant operator, which is written down with respect to a $D \in [D]$, but does not depend on this choice:

$$D_0 : \mathcal{E}^{(ab)}[-2] \mapsto \mathcal{E}^{(ab)}_c{}_0[-2], \tag{5}$$

$$\sigma^{ab} \mapsto D_c \sigma^{ab} - \frac{1}{n+1} \delta_c^{(a} D_p \sigma^{b)p}.$$

D_0 projects the Levi-Civita derivative of a symmetric two tensor σ to its trace-free part. This operator was discussed in [10], where M. Eastwood and V. Matveev showed that this equation governs the metrizability of a projective class of covariant derivatives.

5.1.1 The projective structure as a parabolic geometry

It is a classical result that $(M, [D])$ is equivalent to a unique Cartan geometry (Γ, ω) of type $(G, P) = (\mathrm{SL}(n+1), P)$ with P the stabilizer of a ray in $\mathbb{R}^{n+1}$, see [20],[5].

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1)$ is 1-graded $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathbb{R}^n \oplus \mathfrak{gl}(n) \oplus (\mathbb{R}^n)^*$, where an element $X \oplus (\alpha \,\mathrm{id} + A) \oplus \varphi \in \mathfrak{g}$ for $\alpha \in \mathbb{R}, A \in \mathfrak{sl}(n)$ corresponds to the matrix

$$\begin{pmatrix} -\alpha\frac{n}{n+1} & -\varphi \\ X & \frac{1}{n+1}\alpha\mathbb{I}_n + A. \end{pmatrix}$$

The actions of $\mathfrak{g}_0 = \mathfrak{gl}(n) \subset \mathfrak{g}$ on $\mathfrak{g}_{-1} = \mathbb{R}^n$ and $\mathfrak{g}_1 = (\mathbb{R}^n)^*$ are the standard representation and its dual.

The curvature of the Cartan connection form ω can be regarded as an element of $\mathcal{E}^2(\mathcal{A}M)$, with $\mathcal{A}M = \Gamma \times_P \mathfrak{g}$ the adjoint tractor bundle, and is written

$$K = \begin{pmatrix} 0 & -A_{ac_1c_2} \\ 0 & C_{c_1c_2}{}^a{}_b \end{pmatrix}$$

with A the Cotton-York tensor and C the (projectively invariant) Weyl curvature (cf. [8]).

1-forms and vector fields include into $\mathcal{A}M$ as

$$\eta_a \in T^*M \mapsto \begin{pmatrix} 0 & -\eta_a \\ 0 & 0 \end{pmatrix} \in \mathcal{A}M, \qquad \xi^a \in TM \mapsto \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \in \mathcal{A}M.$$

5.1.2 The operator D_0 as the first BGG-operator

Let $V := \Gamma \times_P S^2\mathbb{R}^{n+1}$. With respect to a choice of Weyl connection $D \in [D]$, a section s of V can be written

$$[s]_D = \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \in \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix} := \begin{pmatrix} \mathcal{E}[-2] \\ \mathcal{E}^a[-2] \\ \mathcal{E}^{(ab)}[-2] \end{pmatrix}. \tag{6}$$

We will need that on the first chain spaces the Lie algebra differentials ∂ and ∂^* are explicitly given by

$$\partial \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab}. \end{pmatrix} = \begin{pmatrix} 0 \\ \rho\delta_c{}^a \\ \delta_c{}^{(a}\mu^{b)} \end{pmatrix} \qquad \partial \begin{pmatrix} \rho_c \\ \mu_c{}^a \\ \sigma_c{}^{ab}. \end{pmatrix} = \begin{pmatrix} 0 \\ 2\delta_{[c_1}{}^a\rho_{c_2]} \\ 2\delta_{[c_1}{}^{(a_1}\mu_{c_2]}{}^{a_2)} \end{pmatrix}.$$

$$\partial^* \begin{pmatrix} \rho_c \\ \mu_c{}^a \\ \sigma_c{}^{ab}. \end{pmatrix} = \begin{pmatrix} -2\mu_p{}^p \\ -2\sigma_p{}^{pa} \\ 0 \end{pmatrix} \qquad \partial^* \begin{pmatrix} \rho_{c_1 c_2} \\ \mu_{c_1 c_2}{}^a \\ \sigma_{c_1 c_2}{}^{ab}. \end{pmatrix} = \begin{pmatrix} 2\mu_{cp}{}^p \\ 2\sigma_{cp}{}^{pa} \\ 0 \end{pmatrix}.$$

As bundles with structure group G_0, V_2, V_1 and $T^*M \otimes V_2$ are irreducible and are contained in the image of ∂^*; $T^*M \otimes V_1$ decomposes into the trace-free part $\operatorname{im}\partial^* \cap T^*M \otimes V_1$ and the trace part, which lies in the image of ∂. The Kostant Laplacian $\square$ acts by

$$\square \begin{pmatrix} \rho_{c_1 c_2} \\ \mu_{c_1 c_2}{}^a \\ \sigma_{c_1 c_2}{}^{ab} \end{pmatrix} = \begin{pmatrix} -2n\rho_{c_1 c_2} \\ -(n+1)\mu_{c_1 c_2}{}^a \\ 0 \end{pmatrix}$$

on V, by multiplication with $-2(n-1)$ on $T^*M \otimes V_2$ and by multiplication with $-n$ on the trace-free part of $T^*M \otimes V_1$. This is all the algebraic information we need to calculate the splitting operators and the prolongation.

The tractor connection ∇^ω on V is easily calculated with the above actions of $\mathcal{E}_a$ and $\mathcal{E}^a$ on V together with the formula $\nabla^\omega = \partial + D + \mathsf{P}\bullet$:

$$\nabla^\omega \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} D_c\rho - 2\mathsf{P}_{ca}\mu^a \\ D_c\mu^a - 2\mathsf{P}_{cb}\sigma^{ab} + \rho\delta_c{}^a \\ D_c\sigma^{ab} + \delta_c{}^{(a}\mu^{b)} \end{pmatrix}.$$

One calculates that the first splitting operator $L_0 : \Gamma(H_0) \mapsto \Gamma(V)$ is given by

$$\sigma^{(ab)} \mapsto \begin{pmatrix} \frac{1}{n(n+1)}D_pD_q\sigma^{pq} + \frac{1}{2n}\mathsf{P}_{pq}\sigma^{pq} \\ -\frac{1}{n+1}D_p\sigma^{pa} \\ \sigma^{ab} \end{pmatrix},$$

and composition of $\nabla^\omega \circ L_0$ with projection to the lowest slot is seen to yield the operator D_0 of (5).

5.1.3 Prolongation of D_0

We calculate the action of the curvature $K \in \Omega^2(M, \mathcal{A}M)$:

$$K_{c_1 c_2} \bullet \begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} -2A_{pc_1 c_2}\mu^p \\ -2A_{pc_1 c_2}\sigma^{pa} + C_{c_1 c_2}{}^a{}_p\mu^p \\ 2C_{c_1 c_2}{}^{(a_1}{}_p\sigma^{a_2)p} \end{pmatrix}. \tag{7}$$

Therefore we define

$$\Phi_1\left(\begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix}\right) := \begin{pmatrix} 0 \\ \bar{\Phi}_1\sigma \\ 0 \end{pmatrix} := -\square^{-1}(\partial^*(K \bullet \begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix})) = \begin{pmatrix} 0 \\ \frac{2}{n}C_{cp}{}^a{}_q\sigma^{pq} \\ 0 \end{pmatrix}.$$

Now the curvature of the modified connection $\nabla^\omega + \Phi_1$ is $R = K \bullet + d^\nabla \Phi_1$ since $(\Phi_1 \wedge \Phi_1)(\xi, \eta)$ vanishes. For $\xi_1, \xi_2 \in \mathfrak{X}(M)$ and $s \in V$

$$(d^\nabla \Phi_1)s(\xi_1, \xi_2) = \tag{8}$$
$$= \nabla_{\xi_1}(\Phi_1(\xi_2)s) - \Phi_1(\xi_2)(\nabla_{\xi_1}s) - \nabla_{\xi_2}(\Phi_1(\xi_1)s) + \Phi_1(\xi_1)(\nabla_{\xi_2}s) - \Phi_1([\xi_1, \xi_2])s.$$

We may expand (8) and write $(d^\nabla \Phi_1)s$ as

$$\left(\begin{pmatrix} \overset{*}{\left(D_{\xi_1}\left(\bar{\Phi}_1(\xi_2)\sigma\right) - \bar{\Phi}_1(\xi_2)\left(D_{\xi_1}\sigma\right) - D_{\xi_2}\left(\bar{\Phi}_1(\xi_1)\sigma\right) + \bar{\Phi}_1(\xi_1)\left(D_{\xi_2}\sigma\right)\right)} \\ -\bar{\Phi}_1([\xi_1, \xi_2])\sigma \\ -\bar{\Phi}_1(\xi_2)\partial_{\xi_1}\varphi + \bar{\Phi}_1(\xi_1)\partial_{\xi_2}\varphi - \bar{\Phi}_1(\xi_2)\partial_{\xi_1}\mu + \bar{\Phi}_1(\xi_1)\partial_{\xi_2}\mu \\ \partial_{\xi_1}\bar{\Phi}_1(\xi_2)\sigma - \partial_{\xi_2}\bar{\Phi}_1(\xi_1)\sigma \end{pmatrix} \right), \tag{9}$$

where we do not take care about the top component since it will vanish after an application of ∂^*. The lowest component is simply $\partial(\bar{\Phi}_1 \sigma) = -\partial\square^{-1}\partial^*(K \bullet \sigma)$. Thus $\partial^*(Rs)$ lies in the top slot (i.e., in homogeneity 1). So our first adjustment had the effect of moving the expression $\partial^*(Rs)$ one slot higher.

The new connection $\nabla^\omega + \Phi_1$ has the following terms in the middle slot of the curvature R_{Φ_1}: ¿From (9) we obtain the terms $2D_{[c_1}\bar{\Phi}_{1c_2]}$ and (via an application of the algebraic Bianchi identity for C), $C_{c_1 c_2 \ p}{}^a \mu^p$. By (7), the contribution of $K \bullet s$ to the middle slot is $-2A_{pc_1c_2}\sigma^{pa} + C_{c_1c_2 \ p}{}^a \mu^p$. In total, we obtain that the action of the curvature R_{Φ_1} is

$$\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \mapsto \begin{pmatrix} \overset{*}{} \\ \frac{2}{n}(D_{[c_1}C_{c_2]p}{}^a{}_q)\sigma^{pq} - 2A_{pc_1c_2}\sigma^{pa} + 2C_{c_1c_2 \ p}{}^a \mu^p \\ * \end{pmatrix}.$$

The entries (*) are irrelevant: the lowest slot is by construction already in the kernel of ∂^* and the highest slot always lies in $\ker \partial^*$. Now define

$$\Phi_2\left(\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \right) := -\square^{-1}\partial^*(R_{\Phi_1}\left(\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \right)).$$

Using $D_p C_{c_1 c_2}{}^p{}_a = (n-2)A_{ac_1c_2}$ and trace-freeness of C, we calculate

$$\Phi_2\left(\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \right) = \begin{pmatrix} -\frac{4}{n}A_{pcq}\sigma^{pq} \\ 0 \\ 0 \end{pmatrix}$$

and obtain that $\Phi := \Phi_1 + \Phi_2 \in \Gamma(T^*M \otimes \mathrm{End}(V))$ is

$$\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \mapsto \frac{2}{n}\begin{pmatrix} -2A_{pcq}\sigma^{pq} \\ C_{cp}{}^a{}_q\sigma^{pq} \\ 0 \end{pmatrix}. \tag{10}$$

Now, with R_Φ the curvature of $\tilde{\nabla} = \nabla^\omega + \Phi$, one has by construction $\partial^* \circ R_\Phi = 0$. Thus $\tilde{\nabla}$ is the prolongation connection for $(D_c \sigma^{ab})_0 = 0$.

5.2 An example in Grassmann geometry

Let $q \in \mathbb{N}, q > 2$ and M be an oriented $2q$-dimensional manifold together with a rank 2-bundle $\mathcal{E}_\alpha$ and a rank q-bundle $\mathcal{E}^{\alpha'}$. Assume there is an isomorphism of TM with $\mathcal{E}_\alpha \otimes \mathcal{E}^{\beta'}$, which will be fixed. We say that M together with the identification $TM = \mathcal{E}_\alpha^{\beta'}$ is a *Grassmannian geometry* of type $(2, q)$ if there exists a torsion-free linear connection D on TM which is the product of linear connections (again denoted by D) on $\mathcal{E}_\alpha$ and $\mathcal{E}^{\beta'}$, see [15], [5]. The class of all such connections are the Weyl connections of $(M, TM \cong \mathcal{E}_\alpha^{\beta'})$.

We are going to prolong the operator

$$D_0 : \mathcal{E}^{[\alpha'\beta']} \mapsto (\mathcal{E}_{\gamma'}^{\gamma[\alpha'\beta']})_0, \tag{11}$$

$$D_0(u^{\alpha'\beta'}) = D_{\gamma'}^\gamma u^{\alpha'\beta'} + \frac{2}{1-q}\delta_{\gamma'}^{[\alpha'} D_{\tau'}^{|\gamma} u^{\tau'|\beta']}.$$

Thus, $D_0(u)$ is the projection of Du to its trace-free part.

5.2.1 Grassmannian structures as parabolic geometries

Let $G = \mathrm{SL}(n)$, $n = 2 + q$ and define P as the stabilizer of a two-plane in $(\mathbb{R}^n)^*$. Regular, normal and torsion-free parabolic geometries $(\mathcal{G}, \omega)$ of type (G, P) are Grassmannian structures. In the Cartan-picture, $\mathcal{E}_\alpha$ and $\mathcal{E}^{\alpha'}$ are associated to the P-representations $(\mathbb{R}^p)^*$, resp. $(\mathbb{R}^q)$.

Let $\mathcal{S}$ be the standard tractor bundle of $(\mathcal{G}, \omega)$, i.e., the associated bundle to the standard representation of $SL(n)$. Via any Weyl structure D, $\mathcal{S}$ decomposes into $(\mathcal{E}^\alpha \oplus \mathcal{E}^{\alpha'})$.

The curvature $K \in \mathcal{E}^2(\mathcal{A}M) = \mathcal{E}^2(\mathcal{S})$ of the Cartan connection is of the form

$$K = \begin{pmatrix} C_{c_1 c_2 \eta}{}^\varphi & -A_{p c_1 c_2} \\ 0 & C'_{c_1 c_2 \eta'}{}^{\varphi'} \end{pmatrix};$$

This employs the (generalized) Weyl curvature components $C \in \Omega^2(M, \mathfrak{sl}(\mathcal{E}^\alpha))$ and $C' \in \mathcal{E}^2(\mathfrak{sl}(\mathcal{E}^{\alpha'}))$ and the generalized *Cotton-York tensor* $A \in \mathcal{E}^2(\mathcal{E}^1)$ (cf. [15]). Normality of the geometry and torsion-freeness imply that any possible trace of $C_{\gamma_1'\gamma_2'\eta}^{\gamma_1\gamma_2\varphi}, C'^{\gamma_1\gamma_2\varphi'}_{\gamma_1'\gamma_2'\eta'}$ and $A^{\varphi\gamma_1\gamma_2}_{\varphi'\gamma_1'\gamma_2'}$ vanishes.

5.2.2 Description of D_0 as first BGG-operator

We consider the tractor bundle $V = \Lambda^2 \mathcal{S}$, which under choice of a Weyl connection D decomposes according to

$$[V]_D = \Lambda^2(\mathcal{E}^\alpha \oplus \mathcal{E}_{\alpha'}) = \begin{pmatrix} \mathcal{E}^{[\alpha\beta]} \\ \mathcal{E}^{\alpha\beta'} \\ \mathcal{E}^{[\alpha'\beta']} \end{pmatrix}.$$

On the first chain spaces the Lie algebra differentials ∂ and ∂^* are given as follows (indices within vertical bars are not included in the skew symmetrization):

$$\partial \begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta^{\beta'}_{\alpha'} v^{\alpha\beta} \\ 2\delta^{[\beta'_1}_{\alpha'} w^{|\alpha|\beta'_2]} \end{pmatrix} \qquad \partial \begin{pmatrix} v^{\gamma\alpha\beta}_{\gamma'} \\ w^{\gamma\alpha\beta'}_{\gamma'} \\ u^{\gamma\alpha'\beta'}_{\gamma'} \end{pmatrix} = \begin{pmatrix} 0 \\ 2\delta^{\beta'}_{\gamma'_2} v^{\gamma_1\gamma_2\beta}_{\gamma'_1} - 2\delta^{\beta'}_{\gamma'_1} v^{\gamma_2\gamma_1\beta}_{\gamma'_2} \\ 2\delta^{[\beta'_1}_{\gamma'_1} w^{|\gamma_2\gamma_1|\beta'_2]}_{\gamma'_2} - 2\delta^{[\beta'_1}_{\gamma'_2} w^{|\gamma_1\gamma_2|\beta'_2]}_{\gamma'_1} \end{pmatrix}$$

$$\partial^* \begin{pmatrix} v^{\gamma\alpha\beta}_{\gamma'} \\ w^{\gamma\alpha\beta'}_{\gamma'} \\ u^{\gamma\alpha'\beta'}_{\gamma'} \end{pmatrix} = \begin{pmatrix} -2w^{[\alpha_1\alpha_2]\tau'}_{\tau'} \\ u^{\alpha\tau'\beta'}_{\tau'} \\ 0 \end{pmatrix} \qquad \partial^* \begin{pmatrix} v^{\gamma_1\gamma_2\alpha\beta}_{\gamma'_1\gamma'_2} \\ w^{\gamma_1\gamma_2\alpha\beta'}_{\gamma'_1\gamma'_2} \\ u^{\gamma_1\gamma_2\alpha'\beta'}_{\gamma'_1\gamma'_2} \end{pmatrix} = \begin{pmatrix} 2w^{\gamma_1[\alpha\beta]\tau'}_{\gamma'_1\tau'} \\ -u^{\gamma_1\alpha\tau'\beta'}_{\gamma'_1\tau'} \\ 0 \end{pmatrix}.$$

The Kostant Laplacian $\square = \partial \circ \partial^* + \partial^* \circ \partial$ acts on $[V]_D$ via

$$\square \begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} (2q)v^{\alpha\beta} \\ (q-1)w^{\alpha\beta'} \\ 0 \end{pmatrix}.$$

The top slot of $\mathcal{E}^1(V)$ is $\mathcal{E}^{[\alpha\beta]}_c = \mathcal{E}^{\gamma[\alpha\beta]}_{\gamma'}$ and coincides with the image of ∂^*. It is irreducible and the Kostant Laplacian acts by multiplication with $2(2q-1)$. The middle slot of $\mathcal{E}^1(V)$, which is $\mathcal{E}^{\alpha\beta'}_c$, decomposes into $\operatorname{im}\partial$, which are traces, and the trace-free part $\operatorname{im}\partial^* = \mathcal{E}^{\alpha\beta'}_{0c}$. One has that $\mathcal{E}^{\gamma\alpha\beta'}_{0\gamma'} = \mathcal{E}^{[\gamma\alpha]\beta'}_{0\gamma'} \oplus \mathcal{E}^{(\gamma\alpha)\beta'}_{0\gamma'}$ and $\square$ acts by q on the alternating part and by $q-2$ on the symmetric part.

The tractor connection on V is

$$(\nabla^\omega)^\gamma_{\gamma'} \begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} D^\gamma_{\gamma'} v^{\alpha\beta} + 2\mathsf{P}^{\gamma[\alpha}_{\gamma'\tau'} w^{\beta]\tau'} \\ D^\gamma_{\gamma'} w^{\alpha\beta'} - \delta^\alpha_{\gamma'} v^{\gamma\beta'} + \mathsf{P}^{\gamma\alpha}_{\gamma'\tau'} u^{\beta'\tau'} \\ D^\gamma_{\gamma'} u^{\alpha'\beta'} + 2\delta^{[\alpha'}_{\gamma'} w^{|\gamma|\beta']} \end{pmatrix}.$$

The first BGG-splitting operator $L_0 : \mathcal{E}^{(\alpha'\beta')} \mapsto \Gamma(V)$ is computed

$$L_0(u^{\alpha'\beta'}) = \begin{pmatrix} \frac{1}{2q}\mathsf{P}^{\alpha\beta}_{\tau'_1\tau'_2} u^{\tau'_1\tau'_2} - \frac{1}{1-q} D^{[\alpha}_{\tau'_1} D^{\beta]}_{\tau'_2} u^{\tau'_1\tau'_2} \\ \frac{1}{1-q} D^\alpha_{\tau'} u^{\tau'\beta'} \\ u^{\alpha'\beta'} \end{pmatrix},$$

and composition of $\nabla^\omega \circ L_0$ with projection to the lowest slot is seen to yield our operator (11).

5.2.3 Prolongation of D_0

For a section s of V one first computes $K \bullet s \in \mathcal{E}^2(V)$, which is then mapped by ∂^* into $\mathcal{E}^1(V)$,

$$\partial^*(K \bullet \begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix}) = \begin{pmatrix} 2C^{\gamma_1[\alpha\beta]}_{\gamma'_1\varphi'\eta} w^{\eta\varphi'} + 2A^{[\alpha|\gamma_1|\beta]}_{\eta'\varphi'} u^{\eta'\varphi'} \\ -2C'^{\gamma_1\alpha\beta'}_{\gamma'_1\varphi'\eta'} u^{\varphi'\eta'} \\ 0 \end{pmatrix}. \tag{12}$$

The first deformation map Φ_1 is defined by $\Phi_1 = -\square^{-1} \circ \partial^* \circ K\bullet$,

$$\Phi_1(\begin{pmatrix} 0 \\ 0 \\ u^{\alpha'\beta'} \end{pmatrix}) = \begin{pmatrix} 0 \\ \frac{2}{q} C'^{[\gamma_1\alpha]\beta'}_{\gamma'_1\varphi'\eta'} u^{\varphi'\eta'} + \frac{2}{q-2} C'^{(\gamma_1\alpha)\beta'}_{\gamma'_1 ph'\eta'} u^{\varphi'\eta'} \\ 0 \end{pmatrix}.$$

Now we need to calculate ∂^* of the change in curvature resulting from Φ_1, which is just $\partial^* \circ d^\nabla \Phi_1$, since one quickly sees that $\partial^* \circ \Phi_{1[c_1}\partial_{c_2]} = 0$. Both indices of a section $w^{\alpha\beta'}$ are contracted into C and the trace taken by ∂^* vanishes by trace-freeness of C, C'. Therefore we are only interested in the differential components of $d^\nabla \Phi_1$ given by

$$
\begin{pmatrix}
0 \\
\begin{pmatrix}
2(\frac{1}{q}+\frac{1}{q-2})D^{\gamma_1}_{\gamma'_1}C'^{\gamma_2\alpha\beta'}_{\gamma'_2\varphi'\eta'}u^{\varphi'\eta'} - 2(\frac{1}{q}-\frac{1}{q-2})D^{\gamma_1}_{\gamma'_1}C'^{\alpha\gamma_2\beta'}_{\gamma'_2\varphi'\eta'}u^{\varphi'\eta'} \\
-2(\frac{1}{q}+\frac{1}{q-2})D^{\gamma_2}_{\gamma'_2}C'^{\gamma_1\alpha\beta'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'} + 2(\frac{1}{q}-\frac{1}{q-2})D^{\gamma_2}_{\gamma'_2}C^{\alpha\gamma_1\beta'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'}
\end{pmatrix} \\
0
\end{pmatrix}.
$$

Applying ∂^* we obtain the top slot contribution

$$
-4(\frac{1}{q}+\frac{1}{q-2})D^{[\alpha}_{\tau'}C'^{|\gamma_1|\beta]\tau'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'} + 4(\frac{1}{q}-\frac{1}{q-2})D^{[\alpha}_{\tau'}C^{\beta]\gamma_1\tau'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'} \tag{13}
$$

Adding the contributions of the top slot of (12) and (13) (after multiplication by $-\frac{1}{2(2q-1)}$) to the modification map Φ_1, we obtain the full modification map

$$
\Phi\begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix}
\frac{1}{2q-1}\begin{pmatrix} 2(\frac{1}{q}+\frac{1}{q-2})D^{[\alpha}_{\tau'}C'^{|\gamma_1|\beta]\tau'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'} - 2(\frac{1}{q}-\frac{1}{q-2})D^{[\alpha}_{\tau'}C^{\beta]\gamma_1\tau'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'} \\ -C^{\gamma_1[\alpha\beta]}_{\gamma'_1\varphi'\eta}w^{\eta\varphi'} + A^{[\alpha|\gamma_1|\beta]}_{\eta'\varphi'}u^{\eta'\varphi'} \end{pmatrix} \\
\begin{pmatrix} \frac{2}{q}C'^{[\gamma_1\alpha]\beta'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'} + \frac{2}{q-2}C'^{(\gamma_1\alpha)\beta'}_{\gamma'_1\varphi'\eta'}u^{\varphi'\eta'} \end{pmatrix} \\
0
\end{pmatrix}.
$$

$\tilde\nabla = \nabla^\omega + \Phi$ is then the prolongation connection of the system $(D^\gamma_{\gamma'}u^{\alpha'\beta'})_0 = 0$.

5.3 The case of infinitesimal automorphisms.

Let $\mathcal{A}M$ be the adjoint tractor bundle of a regular parabolic geometry $(\mathcal{G},\omega)$ over M and ∇^ω the adjoint tractor connection. In [2] it was shown that parallel sections of the connection

$$
\tilde\nabla s = \nabla^\omega s + \kappa(\Pi(s),\cdot) \tag{14}
$$

are in 1-1-correspondence with infinitesimal automorphisms of $(\mathcal{G},\omega)$, where Π is the natural projection $\Pi : \mathcal{A}M \mapsto TM$. It shows that it is interesting to consider the first BGG-operator $\tilde D_0$ obtained by $\tilde\nabla$. If the parabolic geometry $(\mathcal{G},\omega)$ is normal, the curvature of $\tilde\nabla$ lies in the kernel of $\partial^*_{\mathcal{A}M}$. Therefore, exactly as in corollary 3.1, one sees that $\Pi_0 : \mathcal{A}M \mapsto H_0$ and $\tilde L_0 : H_0 \mapsto \mathcal{A}M$ are inverse isomorphisms between the space of parallel sections of $\tilde\nabla$ and the kernel of $\tilde D_0$. Thus, the operator $\tilde D_0$ describes the infinitesimal automorphisms of $(\mathcal{G},\omega)$ and is automatically prolonged by $\tilde\nabla$.

It is shown that if the parabolic geometry is also torsion-free or 1-graded, one has that $\partial^*_{\mathcal{A}M}\kappa = 0$; i.e., for every $s \in \mathcal{A}M$ one has $\partial^*_{\mathcal{A}M}\kappa(\Pi(s),\cdot) = 0$. But in the torsion-free case, the map $\xi \mapsto \kappa(\Pi(s),\xi)$ is evidently homogeneous of degree ≥ 1. Therefore, if we know that $H_1(\mathfrak{g}_-,\mathfrak{g})$ sits in homogeneity ≤ 0, we see that $\xi \mapsto \kappa(\Pi(s),\xi)$ lies in $\operatorname{im}\partial^*_{\mathcal{A}M}$.

Thus we have:

Theorem 5.1 *Let $(\mathcal{G} \mapsto M, \omega)$ be a torsion-free, normal parabolic geometry with $H_1(\mathfrak{g}_-, \mathfrak{g})$ concentrated in homogeneity ≤ 0. Then $\tilde{\nabla}$ from (14) coincides with the normalized tractor connection on $\mathcal{A}M$. In particular, the usual first BGG-operator D_0 coincides with $\tilde{D}_0$ and thus describes infinitesimal automorphisms.*

We note that the homogeneity condition on $H_1(\mathfrak{g}_-, \mathfrak{g})$ is satisfied for all parabolic geometries of type (G, P) with $\mathfrak{g}$ simple and (G, P) not corresponding to projective structures or contact projective structures.

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